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ON STABILITY OF AUTONOMOUS SYSTEMS WITH INTERNAL RESONANCE

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When investigating the stability of the trivial solution of an autonomous system of ordinary differential equations in the critical case of n pairs of pure imaginary roots an essential role can be played by the presence of integral linear dependences between the system's frequencies or, in other words, by the internal resonance. Various special cases of this problem were examined in [1—6]. Our aims are: to obtain a special (normal) form of the differential equation system with internal resonance of most general form in it; to ascertain the conditions under which the presence of internal resonance does not permit the application stability investigation methods developed for resonance-free systems; to solve the stability problem in one of the most important cases of odd-order internal resonance, generalizing the preceding investigations. In the solution of the last problem the necessary and sufficient conditions are given for the stability of the model (simplified) system. Using Chetaev's theorem we show that as a rule the instability of the original system follows from the instability of the model system. Cases of structurally-unstable instability (*) for which the model system does not resolve the problem of stability are outlined. The results obtained are extended, in particular, to Hamiltonian systems.

1. Reduction to normal form. We consider the system of autonomous differential equations

$$x_*^{\cdot} = Ax_* + X_*(x_*) \tag{1.1}$$

$$x_* = (x_1^*, \ldots, x_{2n}^*), \quad X_* = (X_1^*, \ldots, X_{2n}^*), \quad X_*(0) = 0$$

where x_* and X_* are $2n$-dimensional vectors in the Euclidean space E_{2n}, $X_s^*(x_*)$ are holomorphic functions of x_*, A is a constant square matrix with pure imaginary eigenvalues λ_s and $-\lambda_s$ ($\lambda_s^2 < 0$, $s = 1, 2, \ldots, n$), all of which are simple.

Definition. System (1.1) possesses a kth-order internal resonance if relations of form

$$\langle P\Lambda \rangle = 0, \quad \Lambda = (\lambda_1, \ldots, \lambda_n) \tag{1.2}$$

$$P = (p_1, \ldots, p_n), \quad |P| = p_1 + \ldots + p_n = k \geqslant 3, \ P \in E_n$$

are fulfilled between the eigenvalues. Here Λ is the eigenvalue vector of matrix A and the $p_s > 0$ are relatively prime integers.

We restrict consideration to the case when a unique pair of resonance vectors P and Λ exists, i.e. complex resonance is absent (**). Let us construct a nonlinear transforma-

*) Editor's Note. In the Russian text this is called: "noncoarse instability".

**) Certain aspects of the interaction of third-order resonances were considered in [7]; the results of investigating third-order resonance without accounting for interactions, following from [2—4], were also cited therein.

tion reducing the original system to normal form [8] up to terms of arbitrarily high order. To do this we first represent system (1.1) as

$$x^{\cdot} = \lambda x + \sum_{l=m\geqslant 2}^{\infty} X^{(l)}(x, y), \quad y^{\cdot} = -\lambda y + \sum_{l=m\geqslant 2}^{\infty} Y^{(l)}(x, y) \tag{1.3}$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad \lambda = \{\lambda_1, \ldots, \lambda_n\}$$

by means of a nonsingular linear transformation. Here x and y are complex-conjugate vectors, λ is a diagonal matrix, $X^{(l)}$ and $Y^{(l)}$ are complex-conjugate vector functions whose components $X_s^{(l)}$ and $Y_s^{(l)}$ $(s = 1, 2, \ldots, n)$ are represented by lth-order forms.

We transform system (1.3) by the nonlinear substitution

$$x = u + \sum_{l=m}^{2N+1} \Phi^{(l)}(u, v), \quad y = v + \sum_{l=m}^{2N+1} \Psi^{(l)}(u, v)$$

where N is an arbitrarily large number, while $\Phi^{(l)}$ and $\Psi^{(l)}$ are complex-conjugate vector functions whose components $\Phi_s^{(l)}$ and $\Psi_s^{(l)}$ are lth-order forms. Then [9, 10], in the new variables we obtain the following system of equations:

$$u^{\cdot} = \lambda u + \sum_{l=m}^{2N+1} U^{(l)}(u, v) + U(u, v) \tag{1.4}$$

$$v^{\cdot} = -\lambda v + \sum_{l=m}^{2N+1} V^{(l)}(u, v) + V(u, v)$$

in which the expansions of the complex-conjugate vector functions U and V start with terms of not less than $2(N+1)$th order, while $U^{(l)}$ and $V^{(l)}$ are complex-conjugate lth order forms, so that

$$U_s^{(l)}(u, v) = \sum_{|k_s|+|l_s|=l} R_{k_s l_s} u_1^{k_{s1}} \ldots u_n^{k_{sn}} v_1^{l_{s1}} \ldots v_n^{l_{sn}}, \quad s = 1, 2, \ldots, n$$

Here only those coefficients $R_{k_s l_s}$ can be nonzero for which the integral vectors

$$k_s = (k_{s1}, \ldots, k_{sn}), \quad l_s = (l_{s1}, \ldots, l_{sn}), \quad k_{sj}, l_{sj} \geqslant 0$$

satisfy one of the relations

$$\langle (k_s - l_s)\Lambda \rangle = \lambda_s, \quad |k_s| + |l_s| = l, \quad s = 1, 2, \ldots, n \tag{1.5}$$

Such vectors k_s and l_s and the terms in system (1.4) corresponding to them are called resonance vectors and resonance terms, respectively. It is important to ascertain the structure of the resonance terms for solving the stability problem.

As is easily seen, for odd l relations (1.5) are fulfilled identically in λ_s if

$$k_{sj} = l_{sj} + \delta_{sj}, \quad s, j = 1, 2, \ldots n \tag{1.6}$$

where δ_{sj} is the Kronecker symbol. The terms in the lth-order forms, corresponding to these resonance vectors, are called terms of identical resonance. But if Λ satisfies the internal resonance condition (1.2), then relations (1.5) are further satisfied by other values than (1.6) of the vectors k_s and l_s. The terms corresponding to these vectors are called terms of internal resonance. Obviously, relations (1.5) reduce to condition (1.2) in two cases

$$1)\ l_{sj} = \varepsilon p_j + h_{sj} - \delta_{sj}, \quad k_{sj} = h_{sj} \tag{1.7}$$

$$2)\ k_{sj} = \varepsilon p_j + h_{sj} + \delta_{sj}, \quad l_{sj} = h_{sj}$$

where ε is any positive integer, while the h_{sj} are all possible nonnegative integers such that

$$\varepsilon k + 2\sum_{j=1}^{n} h_{sj} = l \pm 1$$

(the plus sign is taken in the first case, while the minus, in the second). From (1.7) it follows that terms of kth-order internal resonance can appear only in forms of not less than ($k-1$)th order. The number ε takes all positive integer values $\varepsilon = 1, 2, \ldots, \varepsilon_*$, where ε_* is the maximum integer contained in the fraction $(l+1)/k$ in the first case and in $(l-1)/k$ in the second.

Thus, to within ($2N+1$)th-order terms the first group of complex-conjugate equations take the form

$$v_s u_s^{\boldsymbol\cdot} = \lambda_s r_s + r_s \sum_{l \geqslant m}^{2N+1} \sum_{2|k_s| = l-1} C_{k_{sj}} \prod_{j=1}^{n} r_j^{k_{sj}} + \tag{1.8}$$

$$\sum_{l=k-1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_1} \prod_{j=1}^{n} v_j^{\varepsilon p_j} \sum_{2|h_s| = l+1-\varepsilon k} C_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_j^{h_{sj}} +$$

$$r_s \sum_{l=k+1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_2} \prod_{j=1}^{n} u_j^{\varepsilon p_j} \sum_{2|h_s| = l-1-\varepsilon k} C_{h_{sj}}^{(\varepsilon)} \prod_{j=1}^{n} r_j^{h_{sj}} + \ldots, \quad r_s = u_s v_s$$

Here the terms left out are of higher than $2(N+1)$th order, while ε_1 and ε_2 are the largest integers contained in the fractions $(l+1)/k$ and $(l-1)/k$, respectively. The first group of nonlinear terms corresponds to identical resonance and is a sum of odd forms starting with degree m or $m+1$ if m is even. The remaining summands are terms of internal resonance in the first and in the second cases (1.7). From examination of (1.8) it follows that the terms of internal resonance do not affect the stability of the trivial solution (under the condition that the problem is solved by the first nonlinear terms) if $k > m+1$ when m is odd and if $k > m+2$ when m is even. Otherwise, the problem must be solved with the resonance terms brought in. Here the solving of the problem is complicated by the fact that it cannot be reduced to the critical case of n zero roots, as can be done in the resonance-free case, i.e. when only terms of identical resonance are present [9—11].

It is convenient to carry out further investigation in the polar coordinates r_s and θ_s

$$u_s = \sqrt{r_s}\, e^{i\theta_s}, \quad v_s = \sqrt{r_s}\, e^{-i\theta_s}, \quad s = 1, \ldots, n$$

Separating the real and imaginary parts in (1.8), we obtain

$$\frac{1}{2} r_s^{\boldsymbol\cdot} = r_s \sum_{l \geqslant m}^{2N+1} \sum_{2|k_s| = l-1} a_{k_{sj}} \prod_{j=1}^{n} r_j^{k_{sj}} + \tag{1.9}$$

$$\sum_{l=k-1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_1} \prod_{j=1}^{n} r_j^{1/2\varepsilon p_j} \sum_{2|h_s| = l+1-\varepsilon k} Q_{h_{sj}}(\varepsilon\theta) \prod_{j=1}^{n} r_j^{h_{sj}} +$$

$$r_s \sum_{l=k+1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_2} \prod_{j=1}^{n} r_j^{1/2\varepsilon p_j} \sum_{2|h_s|=l-1-\varepsilon k} P_{h_{sj}}(\varepsilon\theta) \prod_{j=1}^{n} r_j^{h_{sj}} + \ldots$$

$$\theta^{\cdot} = \sum_{s=1}^{n} p_s \Big[\sum_{l\geqslant m}^{2N+1} \sum_{2|k_s|=l-1} b_{k_{sj}} \prod_{j=1}^{n} r_j^{k_{sj}} +$$

$$\frac{1}{r_s} \sum_{l=k-1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_1} \prod_{j=1}^{n} r_j^{1/2\varepsilon p_j} \sum_{2|h_s|=l+1-\varepsilon k} \frac{dQ_{h_{sj}}(\varepsilon\theta)}{d(\varepsilon\theta)} \prod_{j=1}^{n} r_j^{h_{sj}} -$$

$$\sum_{l=k+1}^{2N+1} \sum_{\varepsilon=1}^{\varepsilon_2} \prod_{j=1}^{n} r_j^{1/2\varepsilon p_j} \sum_{2|h_s|=l-1-\varepsilon k} \frac{dP_{h_{sj}}(\varepsilon\theta)}{d(\varepsilon\theta)} \prod_{j=1}^{n} r_j^{h_{sj}} \Big] + \ldots$$

$$Q_{h_{sj}}(\varepsilon\theta) = a_{h_{sj}}^{(\varepsilon)} \cos\varepsilon\theta + b_{h_{sj}}^{(\varepsilon)} \sin\varepsilon\theta$$

$$P_{h_{sj}}(\varepsilon\theta) = a_{h_{sj}}^{(\varepsilon)} \cos\varepsilon\theta - b_{h_{sj}}^{(\varepsilon)} \sin\varepsilon\theta$$

$$a_{k_{sj}} = \operatorname{Re} c_{k_{sj}},\ a_{h_{sj}}^{(\varepsilon)} = \operatorname{Re} c_{h_{sj}}^{(\varepsilon)},\ b_{k_{sj}} = \operatorname{Im} c_{k_{sj}},\ b_{h_{sj}}^{(\varepsilon)} = \operatorname{Im} c_{h_{sj}}^{(\varepsilon)}$$

If a $2q$th-order system $(q > n)$ of form (1.1) possesses the n frequency resonance (1.2), then the normal form of this system can also be described in form (1.8) or (1.9) with n replaced everywhere by q in the terms of identical resonance. The system obtained from (1.9) by discarding terms of order higher than $2N+1$ in $r_1, \ldots, r_q$ is called a model system. Below we solve the stability problem for system (1.9) in the simplest (and at the same time most important) case of odd-order internal resonance when $k = m+1$.

2. Investigation of the model system. When $k = m + 1$ the model system obtained from (1.9) with $2N + 1 = k$ can be written as follows:

$$r_s^{\cdot} = 2Q_s(\theta) \prod_{j=1}^{n} r_j^{p_j/2}, \quad s = 1, \ldots, n \tag{2.1}$$

$$\theta^{\cdot} = \sum_{s=1}^{n} p_s \frac{dQ_s}{d\theta} \prod_{j=1}^{n} r_j^{p_j/2-\delta_{sj}}, \quad r_j^{\cdot} = 0, \quad j = n+1, \ldots, q$$

$$(\theta = p_1\theta_1 + \ldots + p_n\theta_n, \quad Q_s(\theta) = a_s \cos\theta + b_s \sin\theta)$$

We introduce the matrix

$$C = \begin{Vmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{Vmatrix}$$

with whose aid we set up all possible matrices

$$C_{s_1s_2s_3}' = \begin{Vmatrix} a_{s1} & a_{s2} & a_{s3} \\ b_{s1} & b_{s2} & b_{s3} \end{Vmatrix}, \quad s_1 < s_2 < s_3$$

For each matrix $C_{s_1s_2s_3}$ we set up three determinants

$$D_{s_js_h} = \begin{Vmatrix} a_{s_j} & a_{s_h} \\ b_{s_j} & b_{s_h} \end{Vmatrix}, \quad s_j < s_h$$

By introducing further the auxiliary angles ψ_s

$$\sin\psi_s = -a_s/\Delta_s,\ \cos\psi_s = b_s/\Delta_s,\ \Delta_s = \sqrt{a_s^2 + b_s^2},\ s=1,2,\ldots,n$$

we obtain

$$Q_s(\theta) = \Delta_s \sin(\theta - \psi_s),\quad D_{s_j s_h} = \Delta_{s_j}\Delta_{s_h}\sin(\psi_{s_h} - \psi_{s_j})$$

All the cases which can appear in the investigation of system (2.1) fall into the following groups.

Case 1. a) Rank $C = 2$. A matrix $C_{s_1s_2s_3}$ exists such that all its corresponding determinants $D_{s_js_h} \neq 0$, where

$$\operatorname{sign} D_{s_1s_2} = \operatorname{sign} D_{s_2s_3} = -\operatorname{sign} D_{s_1s_3} \tag{2.2}$$

b) Rank $C = 1$. A pair of elements $a_j,\ a_h \neq 0$ (or $b_j,\ b_h \neq 0$) exist such that $\operatorname{sign} a_ja_h = -1$ or $\operatorname{sign} b_jb_h = -1$).

Case 2. a) Rank $C = 2$. For any matrix $C_{s_1s_2s_3}$ either among the determinants $D_{s_1s_2}, D_{s_1s_3}, D_{s_2s_3}$ there are zero ones or condition (2.2) is violated.

b) Rank $C = 1$. The condition

$$\operatorname{sign} a_ja_h = 1 \quad \text{or} \quad \operatorname{sign} b_jb_h = 1$$

is fulfilled for any nonzero pairs of elements a_ja_h (or $b_j\,b_h$)

Note. Cases 1b and 2b contain the particularly degenerate case when all the determinants $D_{jh} = 0$ ($j,\ h = 1,\ 2,\ \ldots,\ n;\ j \neq h$). In spite of the particularity of this case its investigation is of great interest. In fact, it is not difficult to show that precisely such a degeneracy holds for every Hamiltonian system (*).

The solution of the stability problem in Case 1 is given by

Theorem 2.1. The zero solution of model system (2.1) is stable in Case 1.

Proof. The validity of the theorem follows from the fact that under the conditions of Case 1 system (2.1) possesses the sign-definite integral

$$I \equiv \gamma_1 r_1 + \ldots + \gamma_n r_n + c_{n+1} r_{n+1} + \ldots + c_q r_q = \text{const} \tag{2.3}$$

(c_j are arbitrary constants). Indeed, by computing $dI\,/\,dt$ by virtue of system (2.1), and examining the equation $dI\,/\,dt \equiv 0$, we obtain the following system of equations for determining γ_s:

$$\gamma_1 a_1 + \ldots + \gamma_n a_n = 0, \qquad \gamma_1 b_1 + \ldots + \gamma_n b_n = 0 \tag{2.4}$$

We can convince ourselves that this system has a strictly positive solution when both conditions 1a as well as conditions 1b are fulfilled. The existence of a positive solution proves the theorem.

Going on to consider Case 2, we first state a lemma (omitting the proof) which connects the properties of matrix C and the auxiliary angles ψ_s.

Lemma. When conditions 2a are fulfilled the angles ψ_s can be numbered in such a way as to fulfill the inequalities

$$0 \leqslant \psi_s - \psi_1 < \pi,\quad 0 \leqslant \psi_h - \psi_j < \pi,\quad s = 2,\ldots,\quad n,\quad j < h$$

*) Nurpeisov, S., On stability in the critical case of n pairs of purely imaginary roots in the presence of internal resonance. Alma-Ata, Candidate's Dissertation, 1972.

Taking this numbering of angles ψ_s as accomplished, we number the variables r_s in system (2.1) in the same order. The numbering order is immaterial in the degenerate case since here $\psi_s = \psi_1$ for all s. It is easy to see that as a result of this renumbering the matrix C possesses the property that all the determinants $D_{jh} \geqslant 0$ for $j < h$. In particular, $D_{ns} \leqslant 0$ for all s.

Theorem 2.2. The zero solution of model system (2.1) is unstable in Case 2.

The proof is carried out with the aid of Chetaev's theorem [12]. Obviously, it is sufficient to consider system (2.1) without the equations $r_j^{\cdot} = 0$, $j = n+1, \ldots, q$.

Case 2a. Consider the function

$$V_1 = \prod_{s=1}^{n} r_s^{p_s/2} \cos(\theta - \psi_n)$$

Computing its derivative by virtue of (2.1), after a number of transformations we obtain

$$V_1^{\cdot} = \Delta_n^{-1} \sum_{s=1}^{n-1} p_s D_{sn} \prod_{j=1}^{n} r_j^{p_j - \delta_{js}}$$

whence we see that $V_1^{\cdot} > 0$ in the region $r_s > 0$. Using the equation $V_1 = 0$ we can construct the region $r_s > 0$, $\theta' \leqslant \theta \leqslant \theta''$ in which all the hypotheses of Chetaev's instability theorem are fulfilled.

Case 2b. In this case Chetaev's instability theorem is satisfied either by the function

$$V_2 = \prod_{s=1}^{n} r_s^{p_s/2} \cos\theta$$

(when some of the a_s are nonzero) or by the function

$$V_3 = \prod_{s=1}^{n} r_s^{p_s/2} \sin\theta$$

(when some of the b_s are nonzero).

Theorems 2.1 and 2.2 give the necessary and sufficient stability conditions for the zero solution of the model system (2.1).

In what follows, in the analysis of the complete system we need a more detailed investigation of unstable model systems, which we do below. As we have shown, inequalities (2.5) are fulfilled for every unstable model system. Let us now separate out all the cases of instability into the two classes A and B

$$0 \leqslant \psi_s - \psi_1 < \pi, \quad s = 2, \ldots, n \quad \text{(A)} \tag{2.5}$$

$$0 \leqslant \psi_s - \psi_1 < \pi, \quad s = 2, \ldots, l; \quad \psi_{l+1} = \ldots = \psi_n = \psi_1 + \pi; \quad 2 \leqslant l \leqslant n-1 \quad \text{(B)}$$

Theorem 2.3. Unstable model systems of class A possess a growing solution

$$\theta = \theta_0, \ r_s = \gamma_s z(t), \ \gamma_s > 0, \quad s = 1, 2, \ldots, n \tag{2.6}$$

Substituting this solution into (2.1), we have

$$z^{\cdot} = 2z^{k/2} \tag{2.7}$$

$$\gamma_s = Q_s^{\,l}(\theta) \prod_{j=1}^{n} \gamma_j^{p_j/2}, \quad s = 1, 2, \ldots, n \tag{2.8}$$

$$\sum_{s=1}^{n} \frac{p_s}{\gamma_s} Q_s'(\theta) = 0, \quad Q_s' = \frac{dQ_s}{d\theta} \tag{2.9}$$

First of all, using conditions A we convince ourselves that Eq. (2.9), which we reduce to

$$\sum_{s=1}^{n} p_s \operatorname{ctg}(\theta - \psi_s) = 0$$

by using (2.8), has a root $\theta = \theta_0$ such that $Q_s(\theta_0) > 0$ for all s. Here $\theta_0 \in (\psi_n, \psi_1 + \pi)$ or $\theta_0 \in (\psi_n + \pi, \psi_1)$. Now $\gamma_s > 0$ are determined from (2.8) in the form

$$\gamma_s = Q_{s0}\Big(\prod_{j=1}^{n} Q_{j0}^{p_j}\Big)^{\varkappa}, \quad Q_{s0} = Q_s(\theta_0), \quad \varkappa = \frac{1}{2-k} \tag{2.10}$$

The function $z(t)$ is found from (2.7). By the same token we have given another proof of the instability of the zero solution of the model system when condition A is satisfied.

Theorem 2.4. Unstable model systems of class B possess the sign-constant (positive) integral

$$I \equiv \gamma_1 r_1 + \sum_{s=l+1}^{n} \gamma_s r_s + \sum_{j=n+1}^{q} c_j r_j = \text{const} \tag{2.11}$$

The validity of this assertion can be checked by a direct differentiation of (2.11) on the strength of system (2.1). It is convenient to take as γ_s the following values (c_j are arbitrary constants):

$$\gamma_1 = (n-l)/\Delta_1, \quad \gamma_s = 1/\Delta_s, \quad s = l+1, \ldots, n$$

Note. Model systems of class B do not possess solution (2.6). However, under specific relations between ψ_1 and ψ_j, $j = 2, \ldots, l$, they can have a growing solution analogous to (2.6). For this we need to set in (2.6) $\theta_0 = \psi_1$ or $\theta_0 = \psi_1 + \pi$, $r_s = g_s$, $s = 1, l+1, \ldots, n$, where the g_s are some constants. The existence condition for the growing solution is

$$\sum_{s=2}^{l} p_s \operatorname{ctg}(\psi_1 - \psi_s) = 0$$

3. Investigation of the complete system. The stability of the model system is related with the presence of a sign-definite integral; therefore, it is clear that from its stability it is impossible to draw any conclusions on the stability of the complete system without a consideration of the terms of higher than mth order. However, the following theorem is valid for unstable model systems.

Theorem 3.1. If a model system belongs to class A, then the instability of the zero solution of the model system involves the instability of the zero solution of the complete system.

Setting $2N + 1 = k$, we write system (1.9) as

$$r_s^{\cdot} = 2Q_s(\theta)\prod_{j=1}^{n} r_j^{p_j/2} + R_s(r, \vartheta), \quad s = 1, \ldots, n \tag{3.1}$$

$$\theta^{\cdot} = \sum_{s=1}^{n} p_s Q_s'(\theta) \prod_{j=1}^{n} r_j^{p_j/2-\delta_{js}} + \Phi(r, \Theta)$$

$$r_j^{\cdot} = R_j(r, \Theta), \quad j = n+1, \ldots, q$$

$$r = (r_1, \ldots, r_q),\ \Theta = (\theta_1, \ldots, \theta_q),\ R_s(r, \Theta) \sim O(\|r\|^{(k+1)/2})$$
$$s = 1, \ldots, q;\ \Phi(r, \Theta) \sim O(\|r\|^{(k-1)/2})$$

In (3.1) we pass to the generalized q-dimensional cylindrical coordinates ρ and φ_s, $s = 1, \ldots, n-1$; r_j, $j = n+1, \ldots, q$, by the formulas

$$r_1 = \gamma_1\rho\cos\varphi_1, \quad r_s = \gamma_s\rho\cos\varphi_s\prod_{j=1}^{s-1}\sin\varphi_j, \quad s = 2, \ldots, n-2$$
$$r_n = \gamma_n\rho\prod_{j=1}^{n-1}\sin\varphi_j, \quad r_j = r_j, \quad j = n+1, \ldots, q$$

The following values of angles φ_s:

$$\varphi_s = \varphi_s^\circ, \quad \cos\varphi_s^\circ = (n-s+1)^{-1/2} \quad \sin\varphi_s^0 = \left(\frac{n-1}{n-s+1}\right)^{1/2}$$

correspond to the growing solution in the new coordinate system. We linearize the new system with respect to the variables φ_s and θ in a neighborhood of the point φ_s°, θ_0. Omitting the intermediate computations, we write the final result in the following form:

$$\rho^{\cdot} = 2\omega\rho^{k/2} + F_0(r_*, \alpha, \Theta)$$
$$\alpha_s^{\cdot} = 2\omega\rho^{k/2-1}\left[\frac{LN_s}{(n-s+1)\sqrt{n-1}}\theta_* - \alpha_s\right] + F_s(r_*, \alpha, \Theta), \quad s = 1, \ldots, n-1$$
$$\theta_*^{\cdot} = \omega\rho^{k/2-1}\left(\sum_{s=1}^{n-1}\Gamma_s\alpha_s - k\theta_*\right) + F_n(r_*, \alpha, \Theta)$$
$$r_j^{\cdot} = F_j(r_*, \alpha, \Theta), \quad j = n+1, \ldots, q$$

Here

$$r_* = (\rho, r_{n+1}, \ldots, r_q), \quad \Theta = (\theta_1, \ldots, \theta_n), \quad \alpha = (\alpha_1, \ldots, \alpha_{n-1}), \quad \alpha_s = \varphi_s - \varphi_s^\circ$$
$$\theta_* = \theta - \theta_0, \quad L = \left(\prod_{j=1}^{n}\gamma_j^{p_j}\right)^{1/2}, \quad N_s = \sum_{j=1}^{n-s}\frac{Q'_{s+j,0}}{\gamma_{s+j}} - \frac{n-s}{\gamma_s}Q'_{s0}, \quad \omega = (n^{k/2-1})^{-1/2}$$
$$\Gamma_s = \frac{p_s\sqrt{n-s}}{\gamma_s}Q_{s0}' - \frac{1}{n-s}\sum_{j=s+1}^{n}\frac{p_jQ_{j0}'}{\gamma_j}, \quad \Gamma_{n-1} = -\frac{p_n}{\gamma_n}Q'_{n0}, \quad s=1, \ldots, n-2$$

while functions F_s have the following structure

$$F_s(r_*, \alpha, \Theta) = F_s^{(1)}(r, \alpha, \Theta) + \rho^{(k-1)/2}F_s^{(2)}(r_*, \alpha, \theta_*), \quad s = 1, \ldots, n$$
$$F_s^{(1)} \sim O(\|r\|^{(k-1)/2}), \quad F_s^{(2)}(0, \alpha, \theta_*) \sim O(\|\varphi\|^2, \|\theta_*\|^2)$$
$$F_0(r_*, \alpha, \Theta) = F_0^{(1)}(r_*, \alpha, \Theta) + \rho^{k/2}F_0^{(2)}(r_*, \alpha, \Theta), \quad F_0^{(1)} \sim O(\|r\|^{(k+1)/2})$$
$$F_0^{(2)}(0, \alpha, \Theta) \sim O(\|\alpha\|, \theta_*), \quad F_j(r_*, \alpha, \Theta) \sim O(\|r_*\|^{(k+1)/2}), \quad j = n+1, \ldots, q$$

Having in mind to make use of Chetaev's theorem, we consider the functions

$$V = \rho, \quad W_s = \alpha_s^2 - \rho^{2(1+\gamma)}, \quad s = 1, \ldots, n-1$$
$$W_n = \theta_*^2 - \rho^{2(1+\gamma)}, \quad W_j = r_j - \rho, \quad j = n+1, \ldots, q$$

(γ is a parameter to be defined below). It is clear that the inequality $VV^{\cdot} > 0$ is valid for the derivative $V^{\cdot}$ of the function V for $0 < \| r_* \| < \tau$ (τ is sufficiently small) in some cone K_1 containing the growing solution of the model system. We define a cone K_2 by the inequality

$$\max_s W_s \leqslant 0, \quad s = 1, \ldots, q$$

If γ is sufficiently large, then, obviously, $K_2 \subset K_1$ for all $\rho < \tau$.

We now evaluate the signs of the derivatives $W_{s0}^{\cdot}$ of the functions W_s on their corresponding sections of the surface of the cone K_2, contained in K_1. To this end we set

$$\alpha_j = \delta_j \rho^{1+\gamma}, \ \ \theta_* = \delta_n \rho^{1+\gamma}, \ \ r_i = | \delta_i | \rho, \ \ | \delta_\nu | \leqslant 1$$
$$j = 1, \ldots, \quad n - 1; \quad i = n + 1, \ldots, q; \quad \nu = 1, \ldots, q;$$
$$j \neq i, \ n \neq s$$

If s coincides with one of the values of j, i, s, then we set, respectively $| \alpha_s | = \rho^{1+\gamma}$, $| \theta_* | = \delta_n \rho^{1+\gamma}$, $r_s = \rho$. As a result we obtain

$$W_{s0}^{\cdot} = 4\omega\rho^{\sigma} \Big[\pm \frac{LN_s \delta_n}{(n - s + 1)\sqrt{n - s}} - \gamma - 2 \Big] + O(\rho^{1/2+\sigma}) \tag{3.2}$$
$$W_{n0}^{\cdot} = 2\omega\rho^{\sigma} \Big[\pm \sum_{s=1}^{n} \Gamma_s \delta_s - k - 2(1 + \gamma) \Big] + O(\rho^{1/2+\sigma})$$
$$W_{j0}^{\cdot} = -4\omega(1 + \gamma)\rho^{\sigma} + O(\rho^{1/2+\sigma}), \quad \sigma = 2\gamma + k/2 + 1$$
$$j = 1, \ldots, n - 1; \qquad j = n + 1, \ldots, q$$

From (3.2) we see that for a sufficiently large γ and for all admissible values of δ_ν we have

$$W_{s0}^{\cdot} < 0, \quad s = 1, \ldots, q$$

when $\rho < \tau$. By introducing now the function $W = \max W_s$, $s = 1, \ldots, q$, we can assert that the functions V and W satisfy the corresponding Chetaev's instability theorem [12].

T h e o r e m 3 . 2. If a model system belongs to class B, then the instability of the complete system does not follow from the instability of the model system.

In fact, suppose that system (2.1) is unstable and belongs to class B. We consider the positive definite function

$$H = I + r_2^2 + \ldots + r_l^2 \tag{3.3}$$

where I is the model system's integral (2.11). Setting $R_j(r, \Theta) = 0$, $j = 1, l + 1, \ldots, n$ into system (3.1) and computing the derivative of (3.3), by virtue of system (3.1), we obtain

$$H^{\cdot} = \gamma_1 R_1 + \sum_{j=l+1}^{n} \gamma_j R_j + 4 \sum_{s=2}^{l} r_s Q_s(\theta) \prod_{j=1}^{n} r_j^{p_j/2} \tag{3.4}$$

From (3.4) we see that we can always choose the functions $R_1, R_{l+1}, \ldots, R_n \sim O(\| r \|^{k/2+1})$, so as to have either $H^{\cdot} = 0$, or $H^{\cdot} = G(r, \Theta)$, where $G(r, \Theta)$ is a negative definite function. Q. E. D.

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TIME-OPTIMAL CONTROL SYNTHESIS FOR A FOURTH-ORDER NONLINEAR SYSTEM

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On the basis of Pontriagin's maximum principle we establish the structure of the optimal control and of the optimal trajectories, using the properties of the system being analyzed. We propose a rule for the construction of the program control satisfying the maximum principle. In the case when the terminal state lies outside some bounded region we prove that the rule mentioned determines the optimal control and permits us to solve the synthesis problem.

1. Statement of the problem. Let the motion of a point in the xy-plane be described by the system of equations

$$x^{\cdot} = v \cos \varphi, \quad y^{\cdot} = v \sin \varphi, \quad \varphi^{\cdot} = \frac{K_1}{v} u_1, \quad v^{\cdot} = K_2 u_2 \tag{1.1}$$

where $\varphi = \varphi(t)$ is the angle between the x-axis and the direction of the velocity